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Abstract

The concept of *ellipsoidal anisotropy*, first introduced in linear elasticity by Saint Venant, has reappeared in recent years in diverse applications from the phenomenological to micromechanical modeling of materials. In this concept, *indicator surfaces*, which represent the variation of some elastic parameters in different directions of the material, are ellipsoidal. This concept recovers different models according to the elastic parameters that have ellipsoidal *indicator surfaces*. An interesting feature of some models introduced by Saint Venant is the formation of analytical solutions for basic problems of linear elasticity. This paper has two main objectives. First, an accurate definition of the variety of anisotropy called *ellipsoidal* is provided, which corresponds to a family of materials that depends on 12 independent parameters, including varieties of orthotropic and non-orthotropic materials. An explicit *nondegenerate* Green function solution is established for these materials. Then, it is shown that the *ellipsoidal* model recovers a variety of phenomenological and theoretical models used in recent years, specifically for geomaterials and damaged or micro-cracked materials. These models can be used to approximate the elastic parameters of any anisotropic material with different fitting qualities. A method to optimize the parameters will be given.

Keywords: Linear elasticity, anisotropy, indicator surface, ellipsoidal anisotropy, amorphous materials, damage, Green Function

1. Introduction

In recent years, much research has focused on modeling the elastic anisotropy of noncrystalline materials with simple models. A major focus has been determining adequate models with a reduced number of parameters, which have been constructed using various methods. Cowin and Mehrabadi (1987, 1995) defined classes of elastic materials based on the number and orientation of reflective symmetry planes. This approach was applied to model bone elasticity (Yang *et al.* 1999). Special cases of the fourth order elasticity tensors have also been considered; these cases can be expressed in terms of only one second order tensor. This model has been applied to the micro-mechanical approach of damaged and micro-cracked materials (Halm & Dragon 1988, Dragon *et al.* 2000, Sevostianov & Kachanov 2002, 2008). In other studies, the *indicator surface* properties of some "mono-directional" elastic parameters have been used to define simple forms of elastic anisotropy. A "mono-directional" parameter is a parameter like Young's modulus or the elastic coefficient defined by Eq. (1) that depends on the elasticity tensor $\mathbb{C}$ and only one material direction $\underline{n}$. Examples of mono-directional elastic parameters can be found in Pouya (2007a). The *indicator surface* of a mono-directional parameter is the surface defined by the equation $\underline{x}(\underline{n}) = r(\underline{n})\, \underline{n}$, where $\underline{n}$ scans the unit sphere and represents a direction in the material and $r(\underline{n})$ is the distance from $\underline{x}$ to the origin of coordinates that equals the value of the elastic parameter in the direction $\underline{n}$. The characterization of material anisotropy by the shape of the indicator surfaces was initiated in the early work of Saint Venant (1863), who first introduced the concept of the ellipsoidal anisotropy. Because material isotropy geometrically corresponds to the image of a sphere, anisotropies corresponding to the image of an ellipsoid have naturally been investigated. Saint Venant (1863) studied several elasticity models of this type, arguing the utility of these models in representing the elasticity of anisotropic amorphous materials. This included the study of materials for which the indicator surface of $\sqrt[4]{E(\underline{n})}$, where $E(\underline{n})$ is the Young's modulus in the material direction $\underline{n}$ (see Eq. 31), is an ellipsoid.

However, the research of Saint Venant has previously been neglected in the literature, and the only evidence is a short quote in a book by Lekhnitskii (1963). Independently, the concept of ellipsoidal anisotropy has reappeared in recent years as a guideline for modeling the elasticity of *geomaterials*, such as soils, rocks, and concrete. The anisotropic character of these geomaterials is being accounted for more frequently in different applications, including geotechnical design or the study of seismic wave propagation, genesis of geological structures, and micro-cracking of rocks. A rough representation of the anisotropy with a minimum number of parameters is sufficient for these applications. For this reason, Peres Rodrigues and Aires-Barros (1970) attempted to fit the Young's modulus values of various rocks measured in different directions by ellipsoids. To study seismic wave propagation in geological layers, Daley and Hron (1979) defined the "elliptically anisotropic" medium as being characterized by elliptical P-wave fronts emanating from a point source. Louis *et al.* (2004) proposed a simplified method to analyze the P-wave velocity data in anisotropic rocks; the method assumes an ellipsoidal approximation of certain elastic parameters. Pouya and Reiffsetck (2003) noted that data from Boehler (1975) on the Young's modulus of different soils present an ellipsoidal property, and these authors demonstrated that this assumption allows for the simplification of foundation design.

However, the concept of ellipsoidal anisotropy in the aforementioned works covers a diverse range of materials. The choice of the elastic parameter with ellipsoidal variation leads to very different models. Furthermore, this concept has been used incorrectly in some works. For instance, the Young's modulus, considered by Peres Rodrigues and Aires-Barros (1970), is not an appropriate parameter because its indicator surface can never be an ellipsoid, except when it is a sphere (Pouya 2007a). The second or fourth roots of this parameter should be utilized to define ellipsoidal materials.

The interesting feature of models introduced by Saint Venant is the ability to determine analytical solutions for basic problems in linear elasticity, specifically the Green functions. Therefore, the present study focuses on several special types of ellipsoidal materials introduced by Saint Venant (1863).

This paper has two primary objectives. First, the type of anisotropy called *ellipsoidal* is accurately defined, and an explicit *nondegenerate* expression of the Green function solution is established for these *ellipsoidal* materials. Then, this family of materials is compared to other anisotropic material families existing in the literature, and the *ellipsoidal* family is demonstrated to recover a large variety of models defined for geomaterials and damaged or micro-cracked materials. Finally, it is shown how a given elastic material is approximated by ellipsoidal models followed by a description of the method to obtain the best set of fitting parameters.

Notations:

In the sequel, light-face (Greek or Latin) letters denote scalars, underlined letters denote vectors, bold-faced letters designate second rank tensors or double-index matrices, and outline letters are reserved for fourth rank tensors. The convention of summation on repeated indices is used implicitly. The completely antisymmetric Levi-Civita tensor is denoted ϵ_{ijk} with the components:

$\epsilon_{ijk} = 1$ if i,j,k is an even permutation of 1,2,3

$\epsilon_{ijk} = -1$ if i,j,k is an odd permutation of 1,2,3

$\epsilon_{ijk} = 0$ otherwise.

The scalar product of two vectors $\underline{a}$ and $\underline{b}$ is labeled $\underline{a}.\underline{b}$, and the associated tensor product is $\underline{a}\otimes\underline{b}$ with $(\underline{a}\otimes\underline{b})_{ij} = a_i b_j$. For second rank tensors, the matrix product is labeled $\boldsymbol{AB}$, the inner product is $\boldsymbol{A}:\boldsymbol{B} = A_{ij}B_{ji}$, and the determinant is $|.|$ with (for 3×3 tensors) $|\boldsymbol{a}| = \epsilon_{ijk}\epsilon_{lmn}a_{il}a_{jm}a_{nk}/6$. The operation of the fourth rank tensor $\mathbb{C}$ on $\boldsymbol{A}$ will be labeled as $\mathbb{C}:\boldsymbol{A}$ with $(\mathbb{C}:\boldsymbol{A})_{ij} = C_{ijkl}A_{kl}$ and the operation of $\boldsymbol{A}$ on $\underline{a}$ by $\boldsymbol{A}.\underline{a}$. The tensor transposed from $\boldsymbol{a}$ is denoted $\boldsymbol{a}^{\mathrm{T}}$.

For a fourth rank tensor $\mathbb{C}$ satisfying the symmetries $C_{ijkl} = C_{ijlk} = C_{jikl}$, a *matrix notation* is introduced : the double sub-script (ij) is first abbreviated to a single sub-script (α) running from 1 to 6 by the following rule : 11 → 1, 22 →2, 33 → 3, 23 → 4, 13 → 5, 12 → 6. The *matrix notation* $\boldsymbol{C}$ is defined by the components $c_{\alpha\beta} = C_{(ij)(kl)}$.

2. Ellipsoidal materials

A material with the elasticity tensor $\mathbb{C}$ and a direction in this material indicated by the unit vector $\underline{n}$ will first be considered. The *elastic coefficient* in the direction $\underline{n}$ is defined by:

$$c(\underline{n}) = (\underline{n}\otimes\underline{n}):\mathbb{C}:(\underline{n}\otimes\underline{n}) \qquad (1)$$

The indicator surface of $[c(\underline{n})]^{-1/4}$ is the set of points $\underline{x} = r\,\underline{n}$ with $\underline{n}$ scanning the unit sphere and:

$$r(\underline{n}) = [c(\underline{n})]^{-1/4} \qquad (2)$$

The polynomial equation of this surface is the following:

$$(\underline{x}\otimes\underline{x}):\mathbb{C}:(\underline{x}\otimes\underline{x}) = 1 \qquad (3)$$

Hence, the indicator surface of $[c(\underline{n})]^{-1/4}$ is generally a fourth order surface. For special cases of $\mathbb{C}$, this surface becomes an ellipsoid (second order surface). These cases define a family of materials that we designate by $\hat{\Phi}_4$ and call *ellipsoidal*. If the equation of an ellipsoid is written as $\underline{x}.\boldsymbol{M}.\underline{x} = 1$, then the condition for (3) to define an ellipsoid is the existence of $\boldsymbol{M}$ satisfying:

$$\forall\underline{x};\ (\underline{x}\otimes\underline{x}):\mathbb{C}:(\underline{x}\otimes\underline{x}) = 1 \Leftrightarrow \underline{x}.\boldsymbol{M}.\underline{x} = 1 \qquad (4)$$

Appendix B shows that equation (4) leads to the following general expression for $\mathbb{C}$ where $\boldsymbol{M}$ and $\boldsymbol{L}$ are two second rank symmetric tensors and $\boldsymbol{M}$ is positive-definite:

$$C_{ijkl} = \frac{1}{2}(M_{ik}M_{jl} + M_{il}M_{jk}) + \frac{1}{2}(\epsilon_{ikm}\epsilon_{jln} + \epsilon_{ilm}\epsilon_{jkn})L_{mn} \qquad (5)$$

The family $\hat{\Phi}_4$ depends on 12 independent parameters that are components of $\boldsymbol{M}$ and $\boldsymbol{L}$. Let $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ represent a system of principal axes for the ellipsoid associated to $\mathbb{C}\in\hat{\Phi}_4$. In this coordinate system, $\boldsymbol{M}$ is diagonal. The coefficients of $\boldsymbol{M}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ in this system have the following expressions that are functions of the $c_{\alpha\beta}$ components of $\mathbb{C}$:

$$\boldsymbol{M} = \begin{bmatrix} \sqrt{c_{11}} & & \\ & \sqrt{c_{22}} & \\ & & \sqrt{c_{33}} \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} c_{23} & -c_{36} & -c_{25} \\ -c_{36} & c_{13} & -c_{14} \\ -c_{25} & -c_{14} & c_{12} \end{bmatrix} \tag{6}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & & \\ & c_{22} & c_{23} & & c_{25} & \\ & & c_{33} & & & c_{36} \\ & & & \frac{\sqrt{c_{22}c_{33}}-c_{23}}{2} & -\frac{c_{36}}{2} & -\frac{c_{25}}{2} \\ & & & & \frac{\sqrt{c_{11}c_{33}}-c_{13}}{2} & -\frac{c_{14}}{2} \\ & & & & & \frac{\sqrt{c_{11}c_{22}}-c_{12}}{2} \end{bmatrix} \tag{7}$$

The condition for $\mathbb{C}$ to be positive-definite imposes some conditions on $\boldsymbol{L}$ relative to $\boldsymbol{M}$. These conditions can be expressed more easily by using the following auxiliary tensor $\boldsymbol{L}'$:

$$\boldsymbol{L}' = |\boldsymbol{M}|^{-1}\, \boldsymbol{PLP} \tag{8}$$

where $\boldsymbol{P}$ is the symmetric and positive-definite solution of the equation:

$$\boldsymbol{PP} = \boldsymbol{M} \tag{9}$$

The conditions for $\mathbb{C}$ to be positive-definite are then found to be equivalent to the two following conditions with $\boldsymbol{\delta}$ denoting the second rank unit tensor (see Appendix C):

$$\boldsymbol{\delta} - \boldsymbol{L}'^{2} \text{ positive-definite}, \qquad 1 - \boldsymbol{L}' : \boldsymbol{L}' + 2|\boldsymbol{L}'| > 0 \tag{10}$$

If we take two constants λ and μ and put $\boldsymbol{M} = \sqrt{\lambda + 2\mu}\ \boldsymbol{\delta}$ and $\boldsymbol{L} = \lambda\boldsymbol{\delta}$, then (5) would represent the isotropic material with Lamé coefficients λ and μ. The conditions (10) in this case recover $\mu > 0$ and $3\lambda+2\mu > 0$.

An alternative expression for (5) can be given with $\mathbb{C}$ expressed as a quadratic expression of two second rank tensors. If $|\boldsymbol{L}| \neq 0$, $\varepsilon = \pm 1$ is denoted as the sign of $|\boldsymbol{L}|$ and:

$$\boldsymbol{S} = \sqrt{\varepsilon |\boldsymbol{L}|}\ \boldsymbol{L}^{-1} \tag{11}$$

Then (5) can be written as:

$$C_{ijkl} = \frac{1}{2}(M_{ik} M_{jl} + M_{il} M_{jk}) + \frac{\varepsilon}{2}(2S_{ij} S_{kl} - S_{ik} S_{jl} - S_{il} S_{jk}) \tag{12}$$

This expression is more restricted than (5) because the case ($\boldsymbol{L} \neq 0$ and $|\boldsymbol{L}|=0$) is not recovered, but is useful to demonstrate some properties of the ellipsoidal family as presented in Section 4.4.

Materials defined by these models have two interesting features: 1) recovery of a large family of phenomenological models defined for geomaterials and micro-cracked materials and 2) ability to derive closed-form solutions of Green functions. This latter property is studied in the following section.

3. Green function

The Green function for an infinite medium represents the displacement field for a point force in this medium. It is the key for many basic problems of linear elasticity. An explicit solution of the Green function for $\hat{\Phi}_4$ materials was given by Pouya (2007c). However this solution presents some degeneracy problems which make it inappropriate for numerical modelling. In the following, a nondegenerate expression for this solution will be derived after presenting briefly the method used by Pouya (2007c) to obtain the first solution.

This Green function, denoted by $\boldsymbol{G}(\underline{x})$, is the solution of the equation $C_{ijkl}\partial_{jk}G_{lm}(\underline{x}) = \delta_{im}\delta(\underline{x})$ with the condition $\lim_{\|\underline{x}\|\to\infty} \boldsymbol{G}(\underline{x}) = 0$. This solution can be deduced from the following contour integral (Lifschitz & Rosenweig 1947, Synge 1957, Mura 1982):

$$\boldsymbol{G}(\underline{x}) = \frac{1}{8\pi^2 r}\int_0^{2\pi} \boldsymbol{\Gamma}^{-1}(\underline{n})\, d\theta \qquad (13)$$

In this expression, $r = \|\underline{x}\|$, $\underline{n}(\theta)$ is the unit vector on the circle perpendicular to $\underline{x}$, θ is the polar angle of $\underline{n}$, and $\boldsymbol{\Gamma}(\underline{n})$ is the *acoustic tensor* defined by :

$$\Gamma_{ik} = C_{ijkl}\, n_j n_l \qquad (14)$$

The calculation of $\boldsymbol{G}$ from (13) faces the factorization problem of a polynomial of a degree 6. No general solution is known for this problem. As a matter of fact, $\boldsymbol{\Gamma}^{-1} = \boldsymbol{\Gamma}^{*}/|\boldsymbol{\Gamma}|$, where $\boldsymbol{\Gamma}^{*}$ is the matrix of cofactors of $\boldsymbol{\Gamma}$ and the determinant $|\boldsymbol{\Gamma}|$ is convertible to a polynomial function of $\tan\theta$ that is generally of degree 6. However, Pouya (2007c) demonstrated that the determinant for $\hat{\Phi}_4$ materials can be reduced to a polynomial of degree 2 using the *linear transformation* of the elastic body problem (Pouya 2000, Pouya & Zaoui 2006). For an elastic body with elasticity tensor $\mathbb{C}$ subjected to given surface tractions and displacements, according to this transformation, a simultaneous change of coordinates and of displacement field is considered that is defined by $\underline{x} = \boldsymbol{P}.\, \underline{\tilde{x}}$ and $\underline{u}(\underline{x}) = \boldsymbol{Q}.\, \underline{\tilde{u}}\,(\underline{\tilde{x}})$ where $\boldsymbol{Q} = (\boldsymbol{P}^{T})^{-1}$. The new equations correspond to a new elastic body problem with different geometry, elasticity tensor and prescribed forces and displacements. The transformed elasticity tensor $\tilde{\mathbb{C}}$ is given by:

$$\tilde{C}_{mnpq} = C_{ijkl}Q_{im}Q_{jn}Q_{kp}Q_{lq} \qquad (15)$$

Application of this transformation method to the problem of a point force in an infinite medium allows for the following relation between Green functions $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$ related respectively to $\mathbb{C}$ and $\tilde{\mathbb{C}}$:

$$\tilde{\boldsymbol{G}}\,(\underline{\tilde{x}}) = |\boldsymbol{P}|\, \boldsymbol{P}^{T}\boldsymbol{G}(\underline{x})\boldsymbol{P} \qquad (16)$$

For $\mathbb{C}$ given by (5), let take $\boldsymbol{P}$ equal to the symmetric and positive-definite solution of the equation $\boldsymbol{PP} = \boldsymbol{M}$. The transformation (15) then leads to the following expression for $\tilde{\mathbb{C}}$:

$$\tilde{C}_{ijkl} = \delta_{ij}\,\delta_{kl} - (\in_{ikm}\in_{jln}+\in_{ilm}\in_{jkn})\, T_{mn} \qquad (17)$$

where $\boldsymbol{T}$ is deduced from the following equation in which $\boldsymbol{L}'$ is given by (8):

$$\boldsymbol{L}' = \boldsymbol{\delta} - 2\boldsymbol{T} \qquad (18)$$

The first condition (10) implies that $\boldsymbol{T}$ is positive definite with eigenvalues smaller than 1. We denote $\underline{\tilde{x}} = \boldsymbol{Q}.\underline{x}$, $\tilde{r} = \|\,\underline{\tilde{x}}\|$, $\underline{\hat{x}} = \underline{\tilde{x}}/\tilde{r}$, $\underline{n}$ a unit vector orthogonal to $\underline{\tilde{x}}$ and $\underline{t} = \underline{n}\times\underline{\hat{x}}$ the vector product of $\underline{n}$ and $\underline{\hat{x}}$. The *acoustic tensor* $\tilde{\boldsymbol{\Gamma}}(\underline{n})$ associated to $\tilde{\mathbb{C}}$ has a simple expression with the determinant as follows:

$$|\,\tilde{\boldsymbol{\Gamma}}(\underline{n})| = \xi\alpha - \beta^2 = |\boldsymbol{T}|\,(\underline{n}.\boldsymbol{T}^{-1}.\underline{n}) \qquad (19)$$

with:

$$\xi = \underline{\hat{x}}.\boldsymbol{T}.\underline{\hat{x}}\ , \quad \alpha = \underline{t}.\boldsymbol{T}.\underline{t}\ , \quad \beta = \underline{\hat{x}}.\boldsymbol{T}.\underline{t} \qquad (20)$$

The right-hand side expression in (19) is a second degree trigonometric polynomial in θ, which assures that the line integral (13) for $\tilde{\mathbb{C}}$ can be calculated explicitly. The explicit solution is given by Pouya (2007c) using the eigenvectors of the following tensor $\boldsymbol{B}$:

$$\underline{X} = \boldsymbol{T}.\underline{\hat{x}}\ , \qquad \boldsymbol{B} = \xi\,\boldsymbol{T} - \underline{X}\otimes\underline{X} \qquad (21)$$

The tensor $\boldsymbol{B}$ has two eigenvectors orthogonal to $\underline{\hat{x}}$ with positive eigenvalues that can be written as:

$$\boldsymbol{B} = p^2(\underline{u}\otimes\underline{u}) + q^2(\underline{v}\otimes\underline{v}) \qquad (22)$$

where ($\underline{\hat{x}}$, $\underline{u}$, $\underline{v}$) represents a system of unit eigenvectors of $\boldsymbol{B}$ and $p > 0$ and $q > 0$. With these notations, the solution given by Pouya (2007c) reads:

$$\begin{aligned}
4\pi\,\tilde{r}\,\tilde{\boldsymbol{G}}(\underline{\tilde{x}}) = {} & \frac{\xi}{pq}\,\underline{\hat{x}}\otimes\underline{\hat{x}} + \frac{1}{(p+q)}\left[\frac{1}{p}(\underline{\hat{x}}.\boldsymbol{T}.\underline{u})(\underline{\hat{x}}\otimes\underline{u}+\underline{u}\otimes\underline{\hat{x}}) + \frac{1}{q}(\underline{\hat{x}}.\boldsymbol{T}.\underline{v})(\underline{\hat{x}}\otimes\underline{v}+\underline{v}\otimes\underline{\hat{x}})\right] \\
& + \frac{1}{(p+q)^2}\left[(\underline{u}.\boldsymbol{T}.\underline{u})(\underline{u}\otimes\underline{u}) + (\underline{v}.\boldsymbol{T}.\underline{v})(\underline{v}\otimes\underline{v}) + (\underline{u}.\boldsymbol{T}.\underline{v})(\underline{u}\otimes\underline{v}+\underline{v}\otimes\underline{u})\right] \\
& + \left[\frac{1}{2\,(p+q)^2}(\underline{u}.\boldsymbol{T}.\underline{u}+\underline{v}.\boldsymbol{T}.\underline{v}) + \frac{1}{2}\right](\underline{u}\otimes\underline{u}+\underline{v}\otimes\underline{v}) + \\
& \frac{q-p}{(p+q)^2}\left[\frac{1}{p}(\underline{u}.\boldsymbol{T}.\underline{u})(\underline{u}\otimes\underline{u}) - \frac{1}{q}(\underline{v}.\boldsymbol{T}.\underline{v})(\underline{v}\otimes\underline{v})\right]
\end{aligned} \qquad (23)$$

This expression can be considered an explicit solution for the Green function. However, this solution requires an eigenvalues and eigenvectors calculation to determine (p, q, $\underline{u}$, $\underline{v}$) for each $\underline{x}$ direction. Directions of $\underline{u}$ and $\underline{v}$ become undetermined in certain cases, particularly when $p = q$, and this *degeneracy* causes numerical problems. Accordingly, expression of this solution without referencing $\underline{u}$ and $\underline{v}$ is more appropriate for implementation in numerical programs and for analytical derivations. This can be obtained by determining p and q from the following relations (see Appendix D):

$$p^2 + q^2 = \xi T - \underline{X}.\underline{X}\ , \qquad p^2q^2 = \xi\,\tau \qquad (24)$$

where: $\quad T = \boldsymbol{T} : \boldsymbol{\delta} \qquad \tau = |\boldsymbol{T}| \qquad (25)$

Then, (23) is written as a combination of symmetric expressions in (p, $\underline{u}$)$\leftrightarrow$(q, $\underline{v}$) that can be expressed as a function of $\boldsymbol{\delta}$, $\boldsymbol{T}$, and $\underline{\hat{x}}$. For instance, $(\underline{u}\otimes\underline{u}) + (\underline{v}\otimes\underline{v}) = \boldsymbol{\delta} - \underline{\hat{x}}\otimes\underline{\hat{x}}$ or the equation (22) can be used to eliminate (p, q, $\underline{u}$, $\underline{v}$) in the equations. By using this method and denoting:

$$\eta = (p+q)^2 = \xi T - \underline{X}.\underline{X} + 2\sqrt{\xi\tau} \qquad (26)$$

$$\boldsymbol{F} = -\,\xi\boldsymbol{T} + \left(\eta - \sqrt{\xi\tau}\right)\boldsymbol{\delta} + \underline{X}\otimes\underline{X} + \sqrt{\xi\tau}\;\underline{\hat{x}}\otimes\underline{\hat{x}} \qquad (27)$$

the solution (23) can be written as:

$$\tilde{\boldsymbol{G}}(\tilde{\underline{x}})=\frac{1}{8\pi\tilde{r}}\left\{\left(\frac{T-\xi}{\eta}+1\right)(\boldsymbol{\delta}-\hat{\underline{x}}\otimes\hat{\underline{x}})+\frac{2}{\eta^2\sqrt{\xi\tau}}\boldsymbol{FTF}\right\} \qquad (28)$$

Then, the relation (16) must be used to write the Green function solution for the initial tensor $\mathbb{C}$ in terms of parameters $\boldsymbol{M}$, $\boldsymbol{L}$, and $\underline{x}$. The final result is given in Appendix D.

This solution recovers those obtained for isotropic and Saint Venant materials (see section 4.2). It can be compared to analytical solutions obtained by other methods for special cases of ellipsoidal materials. An interesting comparison is the subfamily of ellipsoidal materials that are transversely isotropic that depend on four intrinsic parameters (see Section 5.1). This comparison can be executed without loss of generality on the corresponding transformed materials $\tilde{\mathbb{C}}$ given by (17). If $\boldsymbol{T} = t_1\boldsymbol{\delta} + (t_3-t_1)\, \underline{n}\otimes\underline{n}$, a material is obtained with transverse isotropy around the axis $\underline{n}$. The solution (28) for this case can then be compared to transversely isotropic materials by Pan & Chou (1976) and Pouya (2007b) with parameters ($c_{11}= c_{22}= c_{33}= 1$, $c_{44} = c_{55} = t_1$, $c_{66} = t_3$). The comparison is very tedious and time-consuming because the two solutions use very different basic functions obtained by fundamentally different ways. Equation (28) is obtained by using line integral (13), while the solution from Pan & Chou (1976) is obtained by displacement potentials method. However, all components of the two solutions were compared and verified as identical.

Notably, several Green function solutions in the literature for different anisotropy cases (Willis 1965, Kroner 1953, Pan & Chou 1976) present degeneracy for special combinations of elastic parameters. A special treatment is then required to adequately derive nondegenerate expressions from these solutions for implementation in numerical codes (Pouya (2007b), Bonnet (2009)). This is also the case of the expression (23) given by Pouya (2007c). However, the solution (28) does not present any *degeneracy*. As previously mentioned, positive definiteness of $\mathbb{C}$ implies p, q >0, and then $\eta > 0$ and $\xi\tau > 0$ according to equations (24) and (26). Therefore, the solution (28) is nondegenerate for every set of elastic parameters.

4. Comparison with other classes of materials

The ellipsoidal family $\hat{\Phi}_4$ covers a large variety of linear elastic materials defined by phenomenological laws or by constitutive relations inspired from micro-mechanical considerations or by other properties. However, this family has restricted intersections with classes of crystalline symmetries. The following section focuses on the possible symmetries of this family and comparison to other families of anisotropic materials.

4.1 Symmetry properties

A geometrical symmetry property of $\mathbb{C}$ necessarily implies the same symmetry for the indicator surface $[c(\underline{n})]^{-1/4}$. Therefore, the possible symmetries of $\mathbb{C}$ may be searched for among the symmetries of the indicator surface. If the half-diameters of an ellipsoid are different, there are only three planes of symmetry and all the possible symmetries reduce to combinations of reflective symmetries relative to these planes. The planes of symmetry for the ellipsoid are determined by the principal directions of $\boldsymbol{M}$. The planes of symmetry for $\boldsymbol{M}$ are not planes of symmetry for $\boldsymbol{L}$ if $\boldsymbol{M}$ and $\boldsymbol{L}$ do not have a common system of principal axes. In this case, these are not planes of symmetry for $\mathbb{C}$ which, therefore, does not have any possible plane of symmetry. Thus, the $\hat{\Phi}_4$ materials cover a variety of non-orthotropic materials that do not have any plane of symmetry. In this case, the Stroh formalism (Stroh 1958, Ting 1996) that establishes closed-form solutions for a large variety of problems in three dimensional anisotropic elasticity cannot be applied because this formalism requires a plane of symmetry for the problem. The elements $\hat{\Phi}_4$ with noncommutable $\boldsymbol{M}$ and $\boldsymbol{L}$ are among the rare variety of non-orthotropic materials with a known closed-form solution of the Green function. Another group of this kind of materials (non-orthotropic with closed-form Green function) was obtained by Pouya (2007b) by application of the *linear transformation* method to transversely isotropic materials.

In contrast, $\mathbb{C}$ is orthotropic when $\boldsymbol{M}$ and $\boldsymbol{L}$ have a common system of principal axes. When there is transverse symmetry for $\boldsymbol{M}$ and $\boldsymbol{L}$ around a common axis, then $\mathbb{C}$ has the same symmetry. Finally, $\mathbb{C}$ is isotropic when $\boldsymbol{M}$ and $\boldsymbol{L}$ are spherical. Therefore, the ellipsoidal anisotropy $\hat{\Phi}_4$ defines, as other families of ellipsoidal anisotropies (Pouya 2007a), a classification that is transversal to other classes of materials defined by rotational invariance or plane symmetries (Forte & Vianello 1996). The symmetries of the indicator surface constitute weaker properties than the symmetries of the actual material. For instance, a spherical indicator surface for $c(\underline{n})$ does not imply that $\mathbb{C}$ is isotropic; it only implies that $\boldsymbol{M}$ is spherical and, in turn, that $\mathbb{C}$ is orthotropic. A more detailed discussion of the relations between the symmetries of the indicator surfaces and the materials is provided in Pouya (2007a).

4.2 Saint Venant material

Let λ and μ represent two Lamé coefficients (μ>0, 3λ+2μ>0) and $\boldsymbol{D}$ a symmetric and positive-definite tensor:

$$\boldsymbol{M} = \sqrt{\lambda+2\mu}\,\boldsymbol{D}\,, \qquad \boldsymbol{L} = \lambda\,|\boldsymbol{D}|\,\boldsymbol{D}^{-1} \tag{29}$$

The relation (5) yields then the following:

$$C_{ijkl} = \lambda D_{ij}D_{kl} + \mu\,(D_{ik}D_{jl} + D_{il}D_{jk}) \tag{30}$$

The Young's modulus in direction $\underline{n}$ is defined by:

$$E(\underline{n}) = \;[(\underline{n}\otimes\underline{n})\!:\!\mathbb{S}:(\underline{n}\otimes\underline{n})]^{-1} \tag{31}$$

where the $\mathbb{S} = \mathbb{C}^{-1}$ represents the compliance tensor. For materials (30), both indicator surfaces of $[c(\underline{n})]^{-1/4}$ and $\sqrt[4]{E(\underline{n})}$ are ellipsoidal. These materials were introduced by Saint Venant (1863) as suitable models for noncrystalline anisotropic materials. Independently, the expression (30) has been used in recent micromechanical studies to represent the effective moduli of heterogeneous media (Milgrom & Shtrikman 1992, Milton 2002) and is suitable to represent the elasticity of geomaterials and some specific rock masses (Chalhoub 2006).

As previously shown by Saint Venant (1863), this family of materials presents interesting theoretical properties. In particular, d'Alembert's displacement potentials and the solution for plane waves propagation in isotropic elasticity can be extended easily to this family. Pouya (2000) demonstrated that this material (30) can be obtained by a *linear transformation* (described herein) from the isotropic material. The theoretical properties of this material called Transformed Isotropic material were investigated by Pouya & Zaoui (2006). Considering the *linear transformation* of the material (30) with $\boldsymbol{P} = \sqrt{\boldsymbol{D}}$, then the transformed $\widetilde{\mathbb{C}}$ will correspond to an isotropic material with Lamé coefficients λ and μ. Therefore, many closed-form solutions for basic elasticity problems with isotropic materials can be extended to *Saint Venant* (30) materials. The following are some examples of results extended to these *Saint Venant*: Eshelby (1957) tensor for inclusion-matrix problem (Milgrom & Shtrikman 1992, Pouya 2000) and the Green function for infinite space (Pouya 2000) and for half-space (Pouya and Zaoui 2006). Extension of the Green function solutions for two joined semi-infinite isotropic solids (Rongved 1955) or for layered medium comprised of isotropic materials (Benitez & Rosakis 1987) to solids constituted of *Saint Venant* materials would also be possible (Pouya and Zaoui 2006).

4.3 Bonnet materials

Bonnet (2009) defined two varieties of orthotropic materials, denoted as CFO2 and CFO4, for which a closed-form solution can be derived for the Green function. These materials are defined on the basis of properties of the sixth order polynomial corresponding to the determinant of the acoustic tensor. As previously described, factorizing this polynomial, which is generally of degree six, to irreducible polynomials of degree less than or equal to four is necessary to determine a closed-form solution for the Green function. Bonnet (2009) introduced two families of orthotropic materials that involve the polynomial being factorized in the product of three second degree polynomials for CFO2 materials and factorized in the product of a second degree and a fourth degree irreducible polynomials for CFO4 materials. Although Bonnet materials are defined on the basis of mathematical considerations, they can provide good approximations for some physical types of materials with well pronounced anisotropy, such as crystals and fiber reinforced composites. In comparison to the $\hat{\Phi}_4$ family, the $\hat{\Phi}_4$ materials are not generally orthotropic, and therefore, are not included in Bonnet materials. However, the determinant $|\tilde{\boldsymbol{\Gamma}}(\underline{n})|$ reduces to a second degree polynomial for a transformed tensor $\tilde{\mathbb{C}}$ of an element $\mathbb{C} \in \hat{\Phi}_4$, which results in $\tilde{\mathbb{C}} \in$ CFO2 and $\tilde{\mathbb{C}} \in$ CFO4. For a general element $\mathbb{C} \in \hat{\Phi}_4$ with $\boldsymbol{P}$ denoting the symmetric and positive solution of $\boldsymbol{P}^2 = \boldsymbol{M}$, consider a unit vector $\underline{n}$ and the unit vector $\underline{\tilde{n}} = (\underline{n}.\boldsymbol{M}.\underline{n})^{-1/2}\,\boldsymbol{P}.\,\underline{n}$. The following can be deduced from equations (14) and (15):

$$\boldsymbol{\Gamma}(\underline{n}) = (\underline{n}.\boldsymbol{M}.\underline{n})\,\boldsymbol{P}\,\tilde{\boldsymbol{\Gamma}}(\underline{\tilde{n}})\boldsymbol{P}$$

Then $|\boldsymbol{\Gamma}(\underline{n})| = (\underline{n}.\boldsymbol{M}.\underline{n})^3\ |\boldsymbol{P}|^2\ |\tilde{\boldsymbol{\Gamma}}(\underline{\tilde{n}})|$. Using the relation (19) for the unit vector $\underline{\tilde{n}}$ and then replacing $\underline{\tilde{n}}$ with $(\underline{n}.\boldsymbol{M}.\underline{n})^{-1/2}\,\boldsymbol{P}.\,\underline{n}$ results in the following:

$$|\boldsymbol{\Gamma}(\underline{n})| = |\boldsymbol{M}|\,|\boldsymbol{T}|\,(\underline{n}.\boldsymbol{M}.\underline{n})^2\ (\underline{n}.\boldsymbol{P}\boldsymbol{T}^{-1}\boldsymbol{P}.\,\underline{n}) \tag{32}$$

Therefore, the $|\boldsymbol{\Gamma}(\underline{n})|$ is factorized in the product of three second degree trigonometric polynomials with two identical, as $\underline{n}.\boldsymbol{M}.\underline{n}$ and $\underline{n}.\boldsymbol{P}\boldsymbol{T}^{-1}\boldsymbol{P}.\underline{n}$ are second degree trigonometric polynomials. This result indicates that orthotropic ellipsoidal materials are a special case of CFO2 materials. However, the Bonnet (2009) materials can easily be extended to a larger family of materials that are not necessarily orthotropic using the *linear transformation* method (Pouya & Zaoui 2006). A transformation tensor $\boldsymbol{P}$ that is not diagonal in the system of orthotropic axes of the CFO2 material is sufficient for this process. From (32), this transformation does not change the nature of $|\boldsymbol{\Gamma}(\underline{n})|$ decomposition in irreducible polynomials. The family of CFO2-extended materials obtained in this way contains $\hat{\Phi}_4$.

4.4 Cracked materials

The "weak anisotropy" can be characterized by the fact that $\boldsymbol{M}$ and $\boldsymbol{L}$ have small non-spherical parts. IF $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ represent two non-spherical tensors with unit Euclidean norm, consider $\boldsymbol{M} = a\,\boldsymbol{\delta} + \alpha\boldsymbol{\omega}_1$ and $\boldsymbol{S} = b\boldsymbol{\delta} + \beta\boldsymbol{\omega}_2$ with $a>0$, $b>0$, $|\alpha/a|<<1$, $|\beta/b|<<1$ and denote the sign of b by $\varepsilon = \pm 1$. The *first order expansion* of (12) with respect to α and β yields:

$$C_{ijkl} \approx \delta_{ij}\mathrm{V}_{kl} + \delta_{kl}\mathrm{V}_{ij} + \delta_{ik}\mathrm{W}_{jl} + \delta_{il}\mathrm{W}_{jk} + \delta_{jk}\mathrm{W}_{il} + \delta_{jl}\mathrm{W}_{ik} \tag{33}$$

with: $\boldsymbol{V} = \varepsilon(b^2\boldsymbol{\delta}+2b\beta\boldsymbol{\omega}_2)$ and $\boldsymbol{W} = [(a^2 - \varepsilon b^2)/4]\boldsymbol{\delta} + (a\alpha\boldsymbol{\omega}_1 - \varepsilon b\beta\boldsymbol{\omega}_2)/2$. Expressions similar to (33) with $\boldsymbol{V}$ and $\boldsymbol{W}$ considered as two independent symmetric tensors have been widely used in various forms to represent the phenomenological model of geomaterial and/or micro-cracked material elasticity. A special case of (33) is obtained by taking two Lamé coefficients λ, μ with $a = \sqrt{\lambda + 2\mu}$, $b = \sqrt{|\lambda|}$, $\varepsilon = \pm 1$ the sign of λ, and $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}$ a symmetric tensor. Then, the expression (33) reads as following:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_1(\delta_{ij}\omega_{kl} + \delta_{kl}\omega_{ij}) + a_2(\delta_{ik}\omega_{jl} + \delta_{il}\omega_{jk} + \delta_{jl}\omega_{ik} + \delta_{jk}\omega_{il}) \tag{34}$$

where $a_1 = 2\varepsilon b\beta$ and $a_2 = (\sqrt{\lambda+2\mu}\,\alpha - \varepsilon b\beta)/2$ are very small. Without assuming small values for (a_1, a_2) compared to (λ,μ), expression (34) has been widely used in the literature to represent the elasticity tensor of damaged materials. From a micromechanical approach using a second order *crack density* tensor $\boldsymbol{\omega}$, Kachanov (1980, 1992) deduced that the effective moduli of an elastic body containing a distribution of cracks can be written as a special case of (34) with $a_1 = 0$ with some approximation. For a material containing a distribution of penny-shaped cracks, the crack density tensor is defined as:

$$\boldsymbol{\omega} = \frac{1}{V}\sum_i \left(r^{(\mathrm{i})}\right)^3 \; \underline{n}^{(\mathrm{i})} \otimes \underline{n}^{(\mathrm{i})} \tag{35}$$

where $r^{(\mathrm{i})}$ and $\underline{n}^{(\mathrm{i})}$ represent the radius and the unit normal to the plane of the disc number i, respectively, and V is the volume of the material. The result obtained by Kachanov (1980, 1992) also contains a fourth-order tensor $\underline{n}^{(\mathrm{i})} \otimes \underline{n}^{(\mathrm{i})} \otimes \underline{n}^{(\mathrm{i})} \otimes \underline{n}^{(\mathrm{i})}$, but the contribution of this tensor was found to be negligible (approximately 10 times less than the other terms) in many current cases, including parallel or isotropic crack arrays (Kachanov 1980). With this approximation, the effective moduli of the cracked material would be given by (34) with λ and μ representing the elastic parameters of the intact material, $a_1 = 0$ and a_2 is a function of (λ, μ).
The same type of expression (34) with $a_1 \neq 0$ could also been obtained for a variety of heterogeneous materials, such as an arbitrary mixture of spheroidal heterogeneities of diverse aspect ratios and orientations (Sevostianov & Kachanov 2002, 2008). The expression (34) is also considered an intermediary between micromechanical and phenomenological models in theoretical investigations on cracked materials (Halm and Dragon 1988, Dragon *et al.* 2000). Finally, this expression has been widely used as a phenomenological model for damaged geomaterials (Chiarelli *et al.* 2000, Alliche 2004).

It is worthy to note that Sevostianov & Kachanov (2008) considered a family of materials with the fourth-rank elasticity tensor $\mathbb{C}$ represented in terms of a symmetric second-rank tensor. These authors termed this family of materials the "*elliptic orthotropy*" materials but did not give an explanation for the adjective *elliptic* or any reference to some properties related to ellipsoids. Expression (34) was determined to be the general expression of the elasticity tensor of these materials. However, the materials (34) can be defined directly by the property that some indicator surface of the material is ellipsoidal, as shown by Pouya (2007). As a matter of fact, the indicator surface of $\left(\sqrt{c(\underline{n})}\right)^{-1}$ is defined in spherical coordinates by the equation $\underline{x} = r\ \underline{n}$ with $r\ (\underline{n}) = \left(\sqrt{c(\underline{n})}\right)^{-1}$, which is equivalent to the polynomial equation $(\underline{x} \otimes \underline{x})\mathbf{:}\mathbb{C}\mathbf{:}(\underline{x} \otimes \underline{x}) = \underline{x}.\ \underline{x}$. The family of *orthotropic* materials with an ellipsoidal indicator surface for $\left(\sqrt{c(\underline{n})}\right)^{-1}$ is the family defined by (34) with arbitrary values, not necessarily small, for a_1 and a_2. The family of materials (not necessarily orthotropic) with an ellipsoidal surface for $\left(\sqrt{c(\underline{n})}\right)^{-1}$ is defined precisely by (33), which can be demonstrated by the methods described in Pouya (2007a, 2007c).

An interesting result of this analysis concerns the family of materials with the elasticity tensor written as (33) with small non-spherical parts for $\boldsymbol{V}$ and $\boldsymbol{W}$ (i.e. $\boldsymbol{V}$ and $\boldsymbol{W}$ can be written as $\boldsymbol{V} = a_v\ \boldsymbol{\delta} + \alpha_v\ \boldsymbol{\omega}_v$ with $\boldsymbol{\omega}_v\mathbf{:}\boldsymbol{\delta} = 0$, $\boldsymbol{\omega}_v\mathbf{:}\boldsymbol{\omega}_v = 1$, $|\alpha_v/a_v| << 1$, and the same relations for $\boldsymbol{W}$). This family includes cases of cracked materials mentioned herein (negligible contribution of the fourth-order tensor) with a weak crack density or damaged materials with an elasticity tensor like (34) with small values for (a_1, a_2). For these materials, an ellipsoidal model (5) can be established as a first order approximation of their elastic behavior. This approximation will allow use of the analytical results obtained herein, particularly the closed-form Green function solutions for the study of these materials.

5. Approximate phenomenological model for materials

The "ellipsoidal" concept and sub-varieties of $\hat{\Phi}_4$ materials have been considered in empirical characterizations of the anisotropic elasticity of soils and rocks. For instance, these have been suitable for a variety of schistose rocks studied by Pinto (1970) and soils studied by Boehler (1975) (see Pouya & Reiffsteck 2003). In this section, approximation of a variety of rocks and rock masses by the ellipsoidal model (5) will be examined, and a general method of fitting the parameters of this model for an arbitrary anisotropic material will be provided.

5.1 *Application to sedimentary rocks and fractured rock masses*

The elastic behavior of most sedimentary rocks can be described by a model of transverse isotropy with an axis of revolution perpendicular to the plane of geological layers. Thomsen (1986) defined four dimensionless parameters, ε, δ, δ^*, and γ, that characterize the transversely isotropic materials and the associated values for a variety of sedimentary rocks. In the context of "weak anisotropy", the objective of Thomsen was to examine the concept of "elliptically anisotropic" medium defined by Daley & Hron (1979) for the study of seismic wave propagation. This concept was widely used in geophysical studies and is different from the concept of ellipsoidal anisotropy considered in the present paper. However, the data from Thomsen (1986) provide the elastic coefficients measured for sedimentary rocks and can be used to examine different approximate models. In this paper, these data are fit to the ellipsoidal model (5). For this purpose, the dimensionless elastic coefficients c^*_{ij} are defined as the ratio $c^*_{ij} = c_{ij}/c_{33}$ with c_{33} representing the elastic coefficient in the direction normal to the geological layers. The values of these coefficients from Thomsen's parameters ε, δ, δ^*, γ are given in Table 1 for rocks studied by Thomsen (1986). The nature of the rock and the depth of the sample are given in the first and second column of the table, respectively, for identification of each material in this table with Thomsen's (1986) table. For an orthotropic material to be ellipsoidal, as deduced from (7), the following three relations have to be satisfied:

$$c_{44} = \frac{\sqrt{c_{22}c_{33}} - c_{23}}{2}, \quad c_{55} = \frac{\sqrt{c_{11}c_{33}} - c_{13}}{2}, \quad c_{66} = \frac{\sqrt{c_{11}c_{22}} - c_{12}}{2} \tag{36}$$

In the context of transverse isotropy, the third relation is automatically satisfied, and the second one is identical to the first relation. Therefore, only one condition must be fulfilled to obtain an ellipsoidal material for a transversely isotropic material (Fig. 1). The distance between a transversely isotropic material and the family of ellipsoidal models can be measured by the difference between the two sides of the first equality in (36) for the real material coefficients. Consequently, a *dimensionless distance d* between the transversely isotropic model and the ellipsoidal model is defined as:

$$d = \frac{1}{c_{33}}\left(c_{44} - \frac{\sqrt{c_{11}c_{33}} - c_{13}}{2}\right) = c^*_{44} - \frac{\sqrt{c^*_{11}} - c^*_{13}}{2} \tag{37}$$

The value of d calculated for the materials in Table 1 is presented in the last column of this table. The parameter c^*_{11} gives an idea of the anisotropy of the initial material (the ratio between c_{11} and c_{33}). For instance, the assumption of an ellipsoidal model induces only a 0.3 % error (d = 0.003) for the clay shale sample at the depth of 5,858.6 m with a noticeable anisotropy c^*_{11}= 1.38. Despite noticeable anisotropy, the other lines of the table demonstrate that the distance to the ellipsoidal model is relatively small. The mean value for d calculated for all sandstone, limestone, mud shale, clay shale, and shale samples (about 25 samples) in the table from Thomsen (1986) is approximately 0.03. Therefore, the ellipsoidal model seems to fit the parameters of these sedimentary rocks.

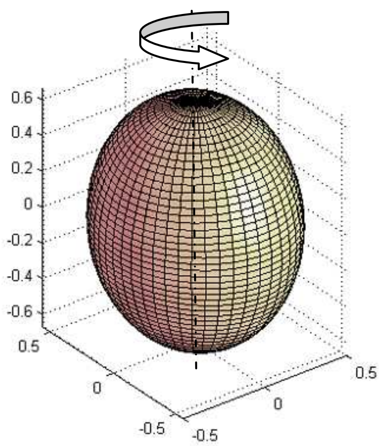

Figure 1: The indicator surface of $[c(\underline{n})]^{-1/4}$ for transversely isotropic material with ellipsoidal anisotropy

Another interesting application of ellipsoidal models is fitting the data obtained by numerical homogenization methods. Numerical homogenization is a common method to determine the properties of cracked or fractured materials, especially rocks and rock masses (Pouya & Ghoreychi 2001, Min & Jing 2003, Chalhoub 2006, Pouya & Chalhoub 2007). In this method, the deformation of a Representative Elementary Volume (REV), which contains a distribution of cracks or fractures stochastically representative of the real material, is simulated under different boundary conditions. Adequate boundary conditions are prescribed to simulate loads in different directions and to determine a complete set of elastic parameters (up to 21) for the anisotropic homogenized material. However, fitting the numerical results with simplified models that contain a reduced number of parameters is generally suitable to simplify data analysis and interpretation. Investigation of this problem for different varieties of rock masses was performed by Chalhoub (2006) in two dimensional plain stress modeling. An example of a limestone sedimentary rock mass containing two families of orthogonal fractures was studied and the indicator surface of $\sqrt[4]{E(\underline{n})}$ was fitted by an ellipsoidal surface. These results demonstrated that the Saint Venant model (30) can satisfactorily fit the numerical results for rock masses containing more than one set of fractures. Moreover, the assumption of ellipsoidal anisotropy proved to be an easy method for estimating the values of out-of-plane elastic parameters, which are not accessible through two dimensional numerical simulation (Chalhoub 2006, Pouya & Chalhoub 2007).

Table 1: Dimensionless parameters for some transversely isotropic sedimentary rocks deduced from Thomsen (1986) data and their distance *d* with ellipsoidal model (Pouya & Chalhoub 2007).

Rock	depth (m)	c^*_{11}	c^*_{44}	c^*_{13}	c^*_{12}	d
Sandstone	4912.0	1.19	0.40	0.28	0.31	-0.004
	5481.3	1.18	0.35	0.44	0.34	0.022
	6542.6	1.16	0.34	0.32	0.36	-0.037
	1582.0	1.16	0.70	-0.34	-0.23	-0.012
Limestone	5469.5	1.11	0.34	0.32	0.34	-0.027
Mudshale	7939.5	1.16	0.33	0.45	0.43	0.019
Clayshale	5501.0	1.67	0.27	0.99	0.49	0.094
	5858.6	1.38	0.30	0.59	0.58	0.003
	3511.0	1.34	0.49	0.02	0.06	-0.069
	450.0	1.22	0.17	0.74	0.76	-0.009
	650.0	1.39	0.17	0.81	0.83	-0.009

5.2 General approximation procedure

The ellipsoidal model (5) can be used to approximate the elastic properties of a general anisotropic material. The quality of fitting will vary for different classes of materials. Therefore, the issue is how to determine the best fitting parameters for a given material, which is in the line of previous research investigating the best material within a given class to approximate a material belonging to a larger class. For example, Pouya & Zaoui (2006) have approximated the elastic properties of different orthotropic crystals by *Saint Venant* materials as defined by (30), and Bonnet (2009) obtained the closest elasticity tensor of CFO2 or CFO4 classes for the same crystals and other orthotropic materials. The following includes a general procedure for determining the best fitting parameters for the model (5).

Approximation of the elastic properties of a given material by a family of models is based on minimization of a *distance* between the elasticity tensor of the material and the family. As mentioned by Bonnet (2009), different *distance*s between two elasticity tensors can be used for this purpose. The properties of these distances have been studied by Moakher & Norris (2006) and Norris (2006). In this work, the Euclidean distance is adopted based upon the Euclidean norm:

$$\| \mathbb{C} \| = \sqrt{C_{ijkl}C_{ijkl}} \tag{38}$$

According to this distance, the best approximate model can be determined within the family of ellipsoidal models (5) for any anisotropic material and applied to crystals.

In the following, $\mathbb{C}$ represents the elasticity tensor of a given material and the associated parameters are known. Obtaining the closest ellipsoidal material to $\mathbb{C}$ consists of determining $\boldsymbol{M}$ and $\boldsymbol{L}$ that minimizes the distance $\| \bar{\mathbb{C}} - \mathbb{C} \|$ where:

$$\bar{C}_{ijkl} = \frac{1}{2}(M_{ik}\, M_{jl} + M_{il}\, M_{jk}) + \frac{1}{2}(\epsilon_{ikm}\epsilon_{jln} + \epsilon_{ilm}\epsilon_{jkn})\, L_{mn} \tag{39}$$

The distance minimization leads to the following system of equations :

$$\partial\|\mathbb{C}-\bar{\mathbb{C}}\|/\partial \boldsymbol{M} = 0 \;\Rightarrow\quad \forall m,n; \quad (C_{ijkl} - \bar{C}_{ijkl})(\delta_{im}\delta_{kn}M_{jl} + \delta_{jm}\delta_{ln}M_{ik}) = 0 \tag{40}$$

$$\partial\| \mathbb{C}-\bar{\mathbb{C}} \|/\partial \boldsymbol{L} = 0 \Rightarrow \quad \forall m,n; \quad (C_{ijkl} - \bar{C}_{ijkl})(\epsilon_{ikm}\epsilon_{jln} + \epsilon_{ilm}\epsilon_{jkn}) = 0 \tag{41}$$

By replacing $\bar{C}_{ijkl}$ in these equations by (39) results in the following system:

$$[(\boldsymbol{M}\mathbf{:}\boldsymbol{M})\, \boldsymbol{M} + \boldsymbol{M}^{3}]_{mn} - L_{ab}\epsilon_{am\alpha}\epsilon_{bn\beta}\, M_{\alpha\beta} = 2\, C_{m\alpha n\beta}\, M_{\alpha\beta} \tag{42}$$

$$L_{mn} = \frac{1}{3}\Big[\, C_{ijkl} - \frac{1}{2}(M_{ik}\, M_{jl} + M_{il}\, M_{jk})\Big]\epsilon_{ikm}\epsilon_{jln} \tag{43}$$

Replacing $\boldsymbol{L}$ in (42) by (43) results in:

$$[(\boldsymbol{M}\mathbf{:}\boldsymbol{M})\, \boldsymbol{M} + 2\boldsymbol{M}^{3}]_{mn} = (C_{mn\alpha\beta} + 2C_{m\alpha n\beta})M_{\alpha\beta} \tag{44}$$

This is a highly nonlinear equation for $\boldsymbol{M}$ that theoretically allows for determination of this tensor. Only a numerical iterative method could be provided to solve this equation. This method consists of beginning with an initial value for $\boldsymbol{M}$, designated as $\boldsymbol{M}^{(0)}$, and then determining $\boldsymbol{M}^{(k+1)}$ from $\boldsymbol{M}^{(k)}$ by the following equation :

$$\boldsymbol{M}^{(k+1)} = \left\{\frac{1}{2}\Big[\boldsymbol{N}^{(k)} - \left(\boldsymbol{M}^{(k)} \mathbf{:} \boldsymbol{M}^{(k)}\right)\boldsymbol{M}^{(k)}\Big]\right\}^{1/3} \tag{45}$$

where :

$$N^{(k)}_{mn} = (C_{mn\alpha\beta} + 2C_{m\alpha n\beta})M^{(k)}_{\alpha\beta} \tag{46}$$

$\boldsymbol{M}^{(0)}$ can be chosen as a function of $\mathbb{C}$ given by (6). This procedure results in issues with convergence and the uniqueness of this solution. However, this procedure has been investigated for some cases of orthotropic materials, which always resulted in a rather quick convergence (less than 1% relative error after approximately 30 iterations).

In the case of orthotropic materials, $\boldsymbol{M}$ and $\boldsymbol{N}$ are diagonal. Denoting the diagonal components of $\boldsymbol{M}$ by (m_1, m_2, m_3), equations (45) and (46) reduce the system to three scalar equations. The first is the following:

$$m_1^{(k+1)} = \left(\frac{1}{2}\Big((3c_{11} - s^{(k)})m_1^{(k)} + (c_{12} + 2c_{66})m_2^{(k)} + (c_{13} + 2c_{55})m_3^{(k)}\Big)\right)^{1/3} \tag{47}$$

and the other two are obtained from this equation by index permutation. In these equations:

$$s^{(k)} = m_1^{(k)} + m_2^{(k)} + m_3^{(k)} \tag{48}$$

The initial values of m_i are taken as $m_1^{(0)} = c_{11}$, $m_2^{(0)} = c_{22}$, $m_3^{(0)} = c_{33}$.

This approximation procedure has been applied to elasticity of some orthotropic, crystalline and composite materials to yield accurate results with a quick convergence. An example of results obtained for the Sulfur crystal is given in Figure 2 and Table 2.

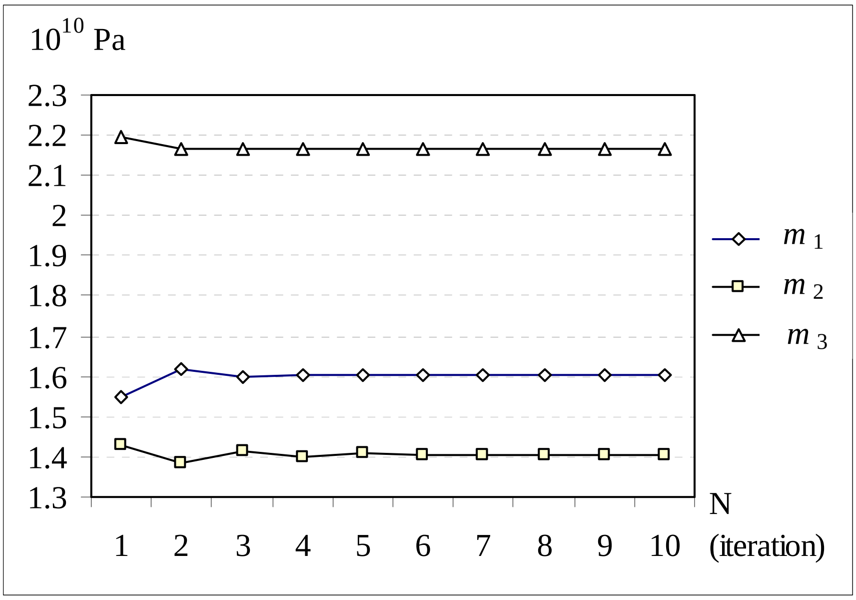

Figure 2 :Iterative process for the Sulfur crystal. The numerical values are given in Table 2.

For this crystal, the relative error ($\|\mathbb{C}^{\mathrm{Appr}} - \mathbb{C}^{\mathrm{M}}\| / \|\mathbb{C}^{\mathrm{M}}\|$) found by the method is less than 0.1. Considering the experimental uncertainty, utilization of the approximate ellipsoidal model can be acceptable for different studies concerning this crystal. Although the global error in Table 2 is less than 0.1, the error on c_{44} is relatively high. This result demonstrates that if the maximum error for individual components of $\mathbb{C}$ are important in some investigations then the Euclidean distance would not be appropriate for optimization of parameters. As previously mentioned, other distances can be chosen for the optimization procedure, but this requires rewriting equations (40) to (48).

Table 2 : Elastic stiffnesses measured for the Sulfur crystal (Dieulesaint and Royer, 1974) and their approximation by the ellipsoidal model. The relative error defined ($\|\mathbb{C}^{\mathrm{Appr}} - \mathbb{C}^{\mathrm{M}}\| / \|\mathbb{C}^{\mathrm{M}}\|$) is less than 0.1.

$c_{\alpha\alpha}$ (10^{10} N/m^2)	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{23}	Relative Error
Measure	2.40	2.05	4.83	0.43	.87	.76	1.33	1.71	1.59	-
Approximation	**2.566**	**1.973**	**4.696**	**0.628**	**0.877**	**0.560**	**1.130**	**1.717**	**1.788**	**0.09895**

Finally, the *Saint Venant* material (30) has four, the *ellipsoidal material* (5) has six, the STrTI model obtained by symmetric transformation of transverse isotropy (Pouya & Zaoui 2006) has six, the CFO2 has seven, and the CFO4 has eight independent parameters in the family of orthotropic materials. The general *ellipsoidal material* (5) (non-orthotropic) has nine *independent parameters* (not accounting for the three Euler angles of the reference system). The precision of the approximation model obtained within these families increases with degrees of freedom for each family.

6. Discussions and conclusions

The concept of ellipsoidal anisotropy is an attractive guideline for modeling the anisotropic elastic behavior of a large family of materials, particularly, soils, rocks, and rock masses. Ellipsoidal anisotropy covers a large variety of models proposed in recent years for geomaterials and cracked or damaged materials obtained by empirical approaches or by micromechanical analyses. The assumption of ellipsoidal anisotropy provides approximate models with a reduced number of parameters allowing for simplification of the data analysis. In addition, ellipsoidal models have very interesting theoretical properties that allow for elaborate closed-form solutions for basic problems of linear elastic bodies. For the general case of ellipsoidal material depending on 12 independent parameters, an explicit and nondegenerate closed-form solution of the Green function has been provided in this paper. This ellipsoidal family covers non-orthotropic materials without any plane of symmetry. The explicit Green function solution for these materials constitutes a rare case of solution not covered by the powerful Stroh (1958) formalism that requires the existence of at least one plane of reflective symmetry. This solution also opens the door to many theoretical and numerical applications, specifically the development of numerical methods using Boundary Elements.

The sub-family of *Saint Venant* materials, which are orthotropic and depend on four independent parameters, offers more facilities for analytical treatments because these materials can be obtained by a *linear transformation* from the isotropic material.

As discussed herein, other hypotheses can also be used to define relatively large families of materials with closed-form solutions of the Green function. The advantage of the ellipsoidal model is the correspondence to anisotropy as a geometrical property and the coverage of a large family of phenomenological, micromechanical and theoretical models in the literature on geomaterials.

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Appendix A : Mathematical identities

The following mathematical identities are useful to demonstrate different results presented in this paper. In particular :

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\ , \qquad \epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij} \tag{A.1}$$

For every second-rank tensor $\boldsymbol{R}$, the following identities can be established:

$$(\boldsymbol{R}^{-1})_{mn} = \frac{1}{2\,|\boldsymbol{R}|}\,\epsilon_{mik}\epsilon_{njl}R_{ij}R_{kl} \tag{A.2}$$

$$\epsilon_{aik}\epsilon_{cjl}R_{ij}R_{kl}R_{cb} = 2\,|\boldsymbol{R}|\,\delta_{ab} \tag{A.3}$$

$$\boldsymbol{R}^{3} = (\boldsymbol{R}\boldsymbol{:}\boldsymbol{\delta})\,\boldsymbol{R}^{2} - \frac{1}{2}\,[(\boldsymbol{R}\boldsymbol{:}\boldsymbol{\delta})^{2} - \boldsymbol{R}\boldsymbol{:}\boldsymbol{R}\,]\,\boldsymbol{R} + |\boldsymbol{R}|\,\boldsymbol{\delta} \tag{A.4}$$

The following result is true if $\boldsymbol{R}$ is symmetric (and can be verified in the reference system where $\boldsymbol{R}$ is diagonal):

$$\epsilon_{amp}\epsilon_{bnq}\epsilon_{ilc}\epsilon_{jkd}\,R_{im}R_{jn}R_{kp}R_{lq}\,R_{c\alpha}\,R_{d\beta} = |\boldsymbol{R}|^{2}\,[\,\delta_{a\alpha}\,\delta_{b\beta} + \delta_{a\beta}\,\delta_{b\alpha}] \tag{A.5}$$

Appendix B : General expression for $\mathbb{C}$

For a positive-definite tensor $\boldsymbol{M}$, the condition $\underline{x}.\boldsymbol{M}.\underline{x} = 1$ is equivalent to $(\underline{x}.\boldsymbol{M}.\underline{x})^2 = 1$. Therefore, the condition (4) in the main text can be written as:

$$\forall \underline{x}\ ;\ \ (\underline{x} \otimes \underline{x})\boldsymbol{:}\mathbb{C}\boldsymbol{:}\,(\underline{x} \otimes \underline{x}) = 1 \Leftrightarrow\ (\underline{x}.\boldsymbol{M}.\underline{x})^{2} = 1 \tag{B.1}$$

The two polynomials are fourth-order and homogeneous. Therefore, this condition is equivalent to:

$$\forall \underline{x}\ ;\ \ (\underline{x} \otimes \underline{x})\boldsymbol{:}\mathbb{C}\boldsymbol{:}\,(\underline{x} \otimes \underline{x}) = (\underline{x}.\boldsymbol{M}.\underline{x})^{2} \tag{B.2}$$

Then, if $\mathbb{F}$ is defined by:

$$F_{ijkl} = C_{ijkl} - \frac{1}{2}\,(M_{ik}\,M_{jl} + M_{il}\,M_{jk}) \tag{B.3}$$

$\mathbb{F}$ has the following properties:

$$\forall \underline{x}\ ; \qquad (\underline{x}\otimes\underline{x})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{x}\otimes\underline{x}) = 0 \tag{B.4}$$

$$\forall\ i,j,k,l; \qquad F_{ijkl} = F_{ijlk} = F_{klij} \tag{B.5}$$

The relation (B.4) implies that for every set of $\underline{x}$, $\underline{y}$, $\underline{a}$, and $\underline{b}$:

$$(\underline{x}\otimes\underline{x})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{x}\otimes\underline{x}) + (\underline{y}\otimes\underline{y})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{y}\otimes\underline{y}) - 2\,(\underline{a}\otimes\underline{a})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{a}\otimes\underline{a}) - 2\,(\underline{b}\otimes\underline{b})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{b}\otimes\underline{b}) = 0 \tag{B.6}$$

Using $\underline{x} = \underline{a}+\underline{b}$, $\underline{y} = \underline{a}-\underline{b}$, and (B.6) results in the following:

$$\forall \underline{a},\ \underline{b}\ ; \qquad (\underline{a}\otimes\underline{a})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{b}\otimes\underline{b}) + 2\,(\underline{a}\otimes\underline{b})\boldsymbol{:}\mathbb{F}\boldsymbol{:}(\underline{a}\otimes\underline{b}) = 0 \tag{B.7}$$

Then if $\mathbb{M}$ is defined by:

$$M_{ijkl} = (F_{ijkl} + F_{ikjl} + F_{iljk})/3 \tag{B.8}$$

This tensor satisfies the symmetries of elasticity tensors $M_{ijkl} = M_{ijlk} = M_{klij}$ and in consequence of (B.7), also satisfies $\forall \underline{a}, \underline{b}$; $(\underline{a}\otimes\underline{a})\boldsymbol{:}\mathbb{M}\boldsymbol{:}(\underline{b}\otimes\underline{b}) = 0$. This is sufficient to state that $\mathbb{M} = 0$. Then, if $\boldsymbol{L}$ is deduced from $\mathbb{F}$ by:

$$L_{pq} = \frac{1}{3}\,\epsilon_{pik}\epsilon_{qjl}F_{ijkl} \tag{B.9}$$

$(\epsilon_{ikm}\epsilon_{jln}+\epsilon_{ilm}\epsilon_{jkn})\, L_{mn}/2 = F_{ijkl} - M_{ijkl}$ using the mathematical identities (A.1), and with $\mathbb{M}=0$:

$$F_{ijkl} = \frac{1}{2}\, (\epsilon_{ikm}\epsilon_{jln}+\epsilon_{ilm}\epsilon_{jkn})\, L_{mn} \tag{B.10}$$

This demonstrates the general expression (5) for $\mathbb{C}$ in the main text.

Appendix C : Properties of $\boldsymbol{L'}$

Applying the transformation (15) on (5) results in the following:

$$\tilde{C}_{mnpq} = \frac{1}{2}(\delta_{mp}\ \delta_{nq} + \delta_{mq}\ \delta_{np}) + H_{mnpq} \tag{C.1}$$

with :$H_{mnpq} = \frac{1}{2}(\epsilon_{ikc}\epsilon_{jld} + \epsilon_{ilc}\epsilon_{jkd})\, Q_{im}Q_{jn}Q_{kp}Q_{lq}L_{cd}$. Defining $\boldsymbol{L}'$ as:

$$L'_{ab} = \frac{1}{6}\, \epsilon_{amp}\epsilon_{bnq}(\epsilon_{ikc}\epsilon_{jld} + \epsilon_{ilc}\epsilon_{jkd})\, Q_{im}Q_{jn}Q_{kp}Q_{lq}L_{cd} \tag{C.2}$$

and using the first equation (A.1), H_{mnpq} can be written as :

$$H_{ijkl} = \frac{1}{2}(\epsilon_{ikm}\epsilon_{jln} + \epsilon_{ilm}\epsilon_{jkn})\, L'_{mn} \tag{C.3}$$

This yields the expression (17) in the main text. If $\boldsymbol{L''} = \boldsymbol{PLP}$ and $\boldsymbol{Q} = \boldsymbol{P}^{-1}$, (C.2) can be written as:

$$L'_{ab} = \frac{1}{6}\, \epsilon_{amp}\epsilon_{bnq}(\epsilon_{ikc}\epsilon_{jld} + \epsilon_{ilc}\epsilon_{jkd})\, Q_{im}Q_{jn}Q_{kp}Q_{lq}\, Q_{c\alpha}\, Q_{d\beta}\, L''_{\alpha\beta} \tag{C.4}$$

The identities (A.5) and (A.1) allow computing:

$\epsilon_{amp}\epsilon_{bnq}\,(\epsilon_{ikc}\epsilon_{jld} + \epsilon_{ilc}\epsilon_{jkd})\, Q_{im}Q_{jn}Q_{kp}Q_{lq}\, Q_{c\alpha}\, Q_{d\beta} = 4|\boldsymbol{Q}|^2\, \delta_{a\alpha}\, \delta_{b\beta} + |\boldsymbol{Q}|^2\, (\delta_{a\alpha}\, \delta_{b\beta} + \delta_{a\beta}\, \delta_{b\alpha})$
and $\boldsymbol{L''}$ is symmetric to determine $\boldsymbol{L'} = |\boldsymbol{Q}|^2\, \boldsymbol{L''}$. Then (9) is used in the main text to find (8).

The condition of $\mathbb{C}$ as positive-definite is equivalent to $\tilde{\mathbb{C}}$, given by (15), to be positive-definite. The expression (17) of this tensor and the reference system with $\boldsymbol{L'}$ diagonal is considered. For an arbitrary symmetric second rank tensor $\boldsymbol{a}$ in this system results in the following:

$$\begin{aligned}\boldsymbol{a} : \tilde{\mathbb{C}} : \boldsymbol{a} = {} & a_{11}^2 + a_{22}^2 + a_{33}^2 + 2L'_{33}\, a_{11}a_{22} + 2L'_{11}\, a_{22}a_{33} + 2L'_{22}\, a_{33}a_{11} \\ & + 2\left[(1-L'_{11})a_{23}^2 + (1-L'_{22})a_{31}^2 + (1-L'_{33})a_{12}^2\right]\end{aligned} \tag{C.5}$$

This is the sum of two independent polynomials in (a_{11}, a_{22}, a_{33}) and (a_{12}, a_{23}, a_{31}). The condition that the first one be positive for every (a_{11}, a_{22}, a_{33}) reads:

$$|L'_{11}| < 1,\quad |L'_{22}| < 1,\quad |L'_{33}| < 1\ ,\qquad L'^2_{11} + L'^2_{11} + L'^2_{11} - 2L'_{11}\, L'_{22}\, L'_{33} < 1 \tag{C.6}$$

These conditions assure that the second polynomial is positive, which can be written in a reference system independent notation as: $\boldsymbol{\delta} - \boldsymbol{L'}^2$ positive-definite and $\boldsymbol{L'}:\boldsymbol{L'} - 2|\boldsymbol{L'}| < 1$.

Appendix D : Calculation of $\boldsymbol{G}$

The method to obtain solution (23) in the main text has been explained in detail in Pouya (2007c). Symmetric combination of these variables should be determined and replaced by functions of $\boldsymbol{\delta}$, $\boldsymbol{T}$, $\underline{\hat{x}}$, etc to eliminate p, q and $\underline{u}$ and $\underline{v}$ in this expression. In particular, $p^2 + q^2 = \boldsymbol{B}:\boldsymbol{\delta} = \xi T - \underline{X}.\underline{X}$. The computation of pq is more technical and involves writing $p^2q^2 = |\boldsymbol{B} + \underline{\hat{x}} \otimes \underline{\hat{x}}|$ based on the expression (22), then using (21) to write $p^2q^2 = |\xi\boldsymbol{T} - \underline{X} \otimes \underline{X} + \underline{\hat{x}} \otimes \underline{\hat{x}}|$ and computing this expression as a function of the components of $\boldsymbol{T}$ in a coordinate system with $\underline{\hat{x}}$ for one axis to determine that $p^2q^2 = \xi|\boldsymbol{T}|$. With these manipulations, (28) is found as a function of the transformed variables. The final expression of **G** is deduced from inversion of (16) and is given in the following.

<u>*Final expression of **G***</u> :

The displacement field in an infinite body with the elasticity tensor (5) defined by ***M*** and ***L*** and subjected to a point force *f* at the origin of coordinates is given by $\underline{U}(\underline{x})=\boldsymbol{G}(\underline{x}).\underline{f}$, where the expression of ***G*** is derived in the following way from ***M***, ***L***, $\underline{x}$, and the unit tensor $\boldsymbol{\delta}$:

P denotes the symmetric and positive solution of $\boldsymbol{P}^2 = \boldsymbol{M}$ and:

$$\boldsymbol{Q} = \boldsymbol{P}^{-1}, \quad \boldsymbol{T} = \frac{1}{2}\left(\boldsymbol{\delta} - |\boldsymbol{M}|^{-1}\boldsymbol{PLP}\right), \quad T = \boldsymbol{T}:\boldsymbol{\delta}, \quad \tau = |\boldsymbol{T}| \qquad \text{(D.1)}$$

$$\tilde{r} = \sqrt{\underline{x}.\boldsymbol{M}^{-1}.\underline{x}}\ , \quad \xi = \frac{1}{\tilde{r}^2}\,\underline{x}.\boldsymbol{QTQ}.\underline{x}, \quad \eta = \xi\,T + 2\sqrt{\xi\tau} - \frac{1}{\tilde{r}^2}\,\underline{x}.\boldsymbol{QT}^2\boldsymbol{Q}.\underline{x} \qquad \text{(D.2)}$$

$$\underline{\hat{x}} = \frac{1}{\tilde{r}}\boldsymbol{Q}.\underline{x}\ , \quad \underline{X} = \boldsymbol{T}.\ \underline{\hat{x}}\ , \quad \boldsymbol{F} = -\xi\boldsymbol{T} + \left(\eta - \sqrt{\xi\tau}\right)\boldsymbol{\delta} + \underline{X}\otimes\underline{X} + \sqrt{\xi\tau}\ \underline{\hat{x}}\otimes\underline{\hat{x}} \qquad \text{(D.3)}$$

Then:

$$\boldsymbol{G}(\underline{x}) = \frac{|\boldsymbol{Q}|}{8\pi\tilde{r}}\boldsymbol{Q}\left\{\left(\frac{T-\xi}{\eta}+1\right)\left(\boldsymbol{\delta} - \underline{\hat{x}}\otimes\underline{\hat{x}}\right) + \frac{2}{\eta^2\sqrt{\xi\tau}}\boldsymbol{FTF}\right\}\boldsymbol{Q} \qquad \text{(D.4)}$$